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Sperner Matroid and Sperner Map

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Abstract. A reconstruction of a Sperner map from a Sperner matroid is illustrated. As an application of this reconstruction, we give a new proof of a completion theorem of Lovász's matroid version of Sperner's lemma.

The purpose of this note is to give a simple procedure which allows us to retrieve a Sperner map from a Sperner matroid.

1. Reconstruction of Sperner map from Sperner matroid

Let K be a triangulation of a d -simplex $a_0a_1 \dots a_d$ in a Euclidean space, and $V(K)$ the vertex set of K . A map $\varphi : V(K) \rightarrow \{0, 1, \dots, d\}$ is said to be a *Sperner map* if for each i_0, i_1, \dots, i_k with $0 \leq i_0 < i_1 < \dots < i_k \leq d$ and for each $v \in V(K) \cap a_{i_0}a_{i_1} \dots a_{i_k}$, $\varphi(v) \in \{i_0, i_1, \dots, i_k\}$. A matroid M on $V(K)$ is called a *Sperner matroid* over K if for each $S \subset \{a_0, a_1, \dots, a_d\}$ and for each $v \in V(K) \cap \text{conv}(S)$, $v \in \text{cl}_M(S)$, where $\text{conv}(S)$ stands for the convex hull of S and $\text{cl}_M(S)$ denotes the closure of S in M . Let M be a Sperner matroid over K such that $\{a_0, a_1, \dots, a_d\}$ forms a basis of M . Put

$$F_j \equiv \text{cl}_M(\{a_0, a_1, \dots, a_j\}) \quad (j = 0, 1, \dots, d).$$

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Let us define $\varphi : V(K) \rightarrow \{0, 1, \dots, d\}$ by setting

$$\begin{aligned}\varphi(v) &= 0 \quad \text{if } v \in F_0, \\ \varphi(v) &= j \quad \text{if } v \in F_j \setminus F_{j-1} \quad (j = 1, 2, \dots, d).\end{aligned}$$

The problem that we consider in this paper is the following : *Under what condition is φ a Sperner map ?* It is clearly necessary that the $(d-1)$ -face $a_1 a_2 \dots a_d$ contains no loops. To see this, if $a_1 a_2 \dots a_d$ contains a loop v of M , then $\varphi(v) = 0$ and $v \in V(K) \cap a_1 a_2 \dots a_d$. As φ is a Sperner map, we have $\varphi(v) \in \{1, 2, \dots, d\}$, in contradiction. What is perhaps surprising is that this condition is also sufficient. Indeed, we have

Theorem 1. *Let M be a Sperner matroid over a triangulation K of a d -simplex $a_0 a_1 \dots a_d$ such that $\{a_0, a_1, \dots, a_d\}$ forms a basis. Put $F_j \equiv cl_M(\{a_0, a_1, \dots, a_j\})$ ($j = 0, 1, \dots, d$), and let $\varphi : V(K) \rightarrow \{0, 1, \dots, d\}$ be defined by*

$$\varphi(v) = 0 \text{ if } v \in F_0, \varphi(v) = j \text{ if } v \in F_j \setminus F_{j-1} \quad (j = 1, 2, \dots, d).$$

Then φ is a Sperner map if and only if the $(d-1)$ -face $a_1 a_2 \dots a_d$ contains no loops of M .

The proof of Theorem 1 is based on the following :

Lemma. *Let B be a basis of a matroid M . Suppose*

- (a) $S \subset T \subset B$,
- (b) $X \subset B$, $X \cap T = \emptyset$,
- (c) $y \in cl_M(T) \setminus cl_M(S)$.

Then $y \in cl_M(T \cup X) \setminus cl_M(S \cup X)$.

2. A sign function

Let K be a triangulation of a d -simplex $a_0 a_1 \dots a_d$ in a Euclidean space, M a Sperner matroid over K , $B \equiv (a_0, a_1, \dots, a_d)$ an ordered basis of M . Let $\Lambda_B : K \rightarrow \{-1, 0, 1\}$. We define $\Lambda_B(v_0 v_1 \dots v_d) = 1$ (resp. -1) if $v_0 \in F_0$ and $v_j \in F_j \setminus F_{j-1}$ ($j = 1, 2, \dots, d$), where $F_j \equiv cl_M(\{v_0, v_1, \dots, v_j\})$ ($j = 0, 1, \dots, d$), and $\det(\alpha_{ij}) > 0$ (resp. $\det(\alpha_{ij}) < 0$), where

$$\begin{pmatrix} v_0 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{0d} \\ \vdots & & \vdots \\ \alpha_{d0} & \cdots & \alpha_{dd} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_d \end{pmatrix}, \quad \sum_{j=0}^d \alpha_{ij} = 1 \quad (0 \leq i \leq d). \quad (*)$$

We define $\Lambda_B(v_0 v_1 \dots v_d) = 0$ otherwise. Now, let $\varphi : V(K) \rightarrow \{0, 1, \dots, d\}$. We call a d -simplex $v_0 v_1 \dots v_d \in K$ *positively* (resp. *negatively*) *completely labelled* if $\varphi(v_j) = j$ ($j = 0, 1, \dots, d$), and $\det(\alpha_{ij}) > 0$ (resp.

$\det(\alpha_{ij}) < 0$), where the matrix (α_{ij}) is given in (*). A d -simplex of K is *completely labelled* if it is positively or negatively completely labelled. It is obvious that $v_0v_1\dots v_d \in K$ is completely labelled if and only if $\{\varphi(v_0), \varphi(v_1), \dots, \varphi(v_d)\} = \{0, 1, \dots, d\}$. The celebrated Sperner lemma[7] asserts that if φ is a Sperner map then $\#\{\sigma \in K ; \sigma \text{ is completely labelled}\} \equiv 1 \pmod{2}$. The oriented Sperner lemma[1] states that

$$\#\{\sigma \in K ; \sigma \text{ is positively completely labelled}\} - \#\{\sigma \in K ; \sigma \text{ is negatively completely labelled}\} = 1.$$

By Theorem 1 and the oriented Sperner lemma, we have

Theorem 2. *Let K be a triangulation of a d -simplex $a_0a_1\dots a_d$ in a Euclidean space, and M a Sperner matroid over K such that the $(d-1)$ -face $a_1a_2\dots a_d$ contains no loops of M . If $B \equiv (a_0, a_1, \dots, a_d)$ is an ordered basis of M , then*

$$\sum_{\sigma \in K} \Lambda_B(\sigma) = 1.$$

Theorem 2 was recently obtained by the authors in [4] with a completely different proof. An example given in [4] shows that the condition “the $(d-1)$ -face $a_1a_2\dots a_d$ contains no loops” cannot be dispensed with.

By disregarding orientation in Theorem 2, we have

Theorem 3. *Under the assumptions of Theorem 2,*

$$\sum_{\sigma \in K} |\Lambda_B(\sigma)| \equiv 1 \pmod{2}.$$

Theorem 3 is an extension of Lovász’s theorem. It complements Lovász’s theorem in two aspects : one concerns an arbitrary matroid while the other is the assertion of oddness. Indeed, Lovász[2] proved the following :

Theorem 4 (L. Lovász). *Let K be a triangulation of a d -simplex $a_0a_1\dots a_d$ in a Euclidean space, and M a Sperner matroid over K such that M contains no loops. If $\{a_0, a_1, \dots, a_d\}$ is a basis of M , then K has a d -simplex $v_0v_1\dots v_d$ such that $\{v_0, v_1, \dots, v_d\}$ is also a basis of M .*

Finally, let us mention that it is perhaps worth developing a general matroid version which contains multiple balanced Sperner lemma[6], combinatorial Lefschetz fixed-point formula[5], and multiple combinatorial Stokes’ theorem[3].

References

1. A. B. Brown and S. S. Cairns, Strengthening of Sperner's lemma applied to homology, Proc. Nat. Acad. Sci. USA, 47(1961), 113-114.
2. L. Lovász, Matroids and Sperner's lemma, Europ. J. Combinatorics, 1 (1980), 65-66.
3. S. N. Lee and M. H. Shih, A counting lemma and multiple combinatorial Stokes' theorem, Europ. J. Combinatorics, 19(1998), 969-979.
4. S. N. Lee and M. H. Shih, Sperner matroid, Arch. der Math. to appear.
5. M. H. Shih and S. N. Lee, A combinatorial Lefschetz fixed-point formula, J. Combinatorial Theory, Ser. A, 61(1992), 123-129.
6. M. H. Shih and S. N. Lee, Combinatorial formulae for multiple set-valued labellings, Math. Ann. 296(1993), 35-61.
7. E. Sperner, Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes, Abh. Math. Sem. Univ. Hamburg, 6 (1928), 265-272.